Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Geometry and Physics 50 (2004) 393–414
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Differential operators on a Riemann surface with projective structure

Indranil Biswas*

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

Received 26 March 2003; received in revised form 17 July 2003

Abstract

Let X be a Riemann surface equipped with a projective structure $\mathfrak{p}$ and $\mathcal{L}$ a theta characteristic on X, or in other words, $\mathcal{L}$ is a holomorphic line bundle equipped with a holomorphic isomorphism with the holomorphic cotangent bundle Ω_X. The complement of the zero section in the total space of the line bundle $\mathcal{L}$ has a natural holomorphic symplectic structure, and using $\mathfrak{p}$, this symplectic structure has a canonical quantization. Using this quantization, holomorphic differential operators on X are constructed. The main result is the construction of a canonical isomorphism

$$H^0(X, \mathrm{Diff}_X^k(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)})) \cong \bigoplus_{l=0}^{k} H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega_X^{\otimes l}),$$

$i, j \in \mathbb{Z}$, $n \geq 0$, provided $i \notin [-2(k-1), 0]$.

JGP SC: Quantum mechanics

MSC: 14F05; 32C38; 53B10

Keywords: Differential operator; Projective structure; Quantization

1. Introduction

Let X be a Riemann surface, not necessarily compact or of finite type. Fix a holomorphic line bundle $\mathcal{L}$ over X together with a holomorphic isomorphism of $\mathcal{L}^{\otimes 2}$ with the holomorphic cotangent bundle Ω_X. In other words, $\mathcal{L}$ is a theta characteristic on X.

A projective structure on X is a covering of X by holomorphic coordinate functions such that all the transition functions are Möbius transformations. (Möbius transformations

* Fax: +91-22-2280-4610.
E-mail address: indranil@math.tifr.res.in (I. Biswas).

0393-0440/$ – see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2003.09.002

are functions of the form $z \mapsto (az+b)/(cz+d), a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$; so the group defined by all Möbius transformations is identified with $\mathrm{PSL}(2, \mathbb{C})$.) Every Riemann surface admits a projective structure, and the space of all projective structures on X is an affine space for $H^0(X, \Omega_X^{\otimes 2})$. Given a projective structure on X, using the theta characteristic $\mathcal{L}$ the transition functions can be lifted from the Möbius group $\mathrm{PSL}(2, \mathbb{C})$ to $\mathrm{SL}(2, \mathbb{C})$ satisfying the cocycle condition (see Section 2.1 for the details).

Let $\mathcal{Z}$ denote the complex surface defined by the complement of the zero section in the total space of the line bundle $\mathcal{L}$. The complex manifold $\mathcal{Z}$ has a holomorphic symplectic structure induced by the standard symplectic form on the total space of Ω_X. More precisely, the symplectic structure on $\mathcal{Z}$ is the pullback, using the map $v \mapsto v \otimes v$, of the standard symplectic form on the total space of Ω_X.

In [3] it was shown that for each projective structure on X there is a canonically associated quantization of this symplectic surface $\mathcal{Z}$. Let $\mathcal{H}(\mathcal{Z})$ denote the space of all (locally defined) holomorphic functions on $\mathcal{Z}$. We recall that a quantization is an associative multiplication operation

$$\star : \mathcal{H}(\mathcal{Z}) \otimes_{\mathbb{C}} \mathcal{H}(\mathcal{Z}) \to \mathcal{H}(\mathcal{Z})[[h]]$$

with h being a formal parameter, such that for $h = 0$ it is the pointwise product on $\mathcal{H}(\mathcal{Z})$, and the derivative at $h = 0$ of this $\star$ operation is given by the Poisson structure on $\mathcal{H}(\mathcal{Z})$ defined by the symplectic form (see Section 2.2 for the details).

Fix a projective structure $\mathfrak{p}$ on X. Consequently, we have a quantization of the symplectic variety $\mathcal{Z}$.

A holomorphic section

$$s \in H^0(X, \mathcal{L}^{\otimes i})$$

defines a holomorphic function on $\mathcal{Z}$, which will be denoted by Γ_s. For any $z \in \mathcal{Z}$ projecting to $x \in X$, if $s(x) = cz^{\otimes i}$, then $\Gamma_s(z) = c$ (see Section 3). For another holomorphic section $t \in H^0(X, \mathcal{L}^{\otimes j})$, let

$$\Gamma_s \star \Gamma_t = \sum_{k=0}^{\infty} h^k \Psi_k$$

be the quantization product.

We observe that for any $k \geq 0$, there is a (unique) section

$$u \in H^0(X, \mathcal{L}^{\otimes (i+j+2k)})$$

such that $\Gamma_u = \Psi_k$ (Lemma 3.1).

As a consequence of Lemma 3.1, fixing the section s we get a unique holomorphic differential operator on X

$$S_s^k(j) \in H^0(X, \mathrm{Diff}_X^k(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes (i+j+2k)}))$$

which is determined by the following two conditions:

1. for any section t of $\mathcal{L}^{\otimes j}$, its evaluation $S_s^k(j)(t)$ coincides with the section u given by Lemma 3.1;

2. for any open subset U of X equipped with the projective structure induced by $\mathfrak{p}$, the differential operator $S^k_s(j)_U$ obtained by substituting X by U coincides with the restriction of $S^k_s(j)$ to U (that is, $S^k_s(j)$ is a local operator).

The symbol $\sigma(S^k_s(j))$ of the differential operator $S^k_s(j)$ is

$$\sigma(S^k_s(j)) = \binom{i+k-1}{k}\left(\frac{\sqrt{-1}}{2}\right)^k s$$

(see Lemma 4.1).

Using Lemma 4.1 inductively, it is possible, for suitable values of i and k, to decompose, in a canonical fashion, a differential operator

$$D \in H^0(X, \mathrm{Diff}^k_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)}))$$

into a sum of $k+1$ differential operators of order $0, 1, \dots, k$. Note that in general only the symbol of a differential operator makes sense; subsequent order terms do not make sense in general.

More precisely, if $i \notin [-2(k-1), 0]$, we have an isomorphism

$$\bigoplus_{l=0}^{k} H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega_X^{\otimes l}) \to H^0(X, \mathrm{Diff}^k_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)}))$$

that sends any $s_l \in H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega_X^{\otimes l})$, where $l \in [0, k]$, to the differential operator $S^{k-l}_{s_l}(j)$ given by Lemma 4.1 (Theorem 5.1). Furthermore, the image of $H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega_X^{\otimes l})$, by the above homomorphism, is contained in

$$H^0(X, \mathrm{Diff}^{k-l}_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)})) \subset H^0(X, \mathrm{Diff}^k_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)}))$$

for each $l \in [0, k]$. In other words, the above decomposition of a differential operator into sections of $H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega_X^{\otimes l})$, $l \in [0, k]$, is compatible with the filtration of $H^0(X, \mathrm{Diff}^{k-l}_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)}))$ defined by the lower order differential operators.

The space of all differential operators of order k from $\mathcal{L}^{\otimes j}$ to $\mathcal{L}^{\otimes(i+j+2k)}$ has a canonical filtration which is given by operators of order l with $l \in [0, k]$. However, Theorem 5.1 says that after choosing a projective structure on X, this filtration of differential operators has a natural semisimplification.

This decomposition extends to differential operators on $W \otimes \mathcal{L}^{\otimes j}$, where W is a vector bundle over X equipped with a holomorphic connection (see Remark 5.2).

2. Preliminaries

2.1. Projective structure

Take a complex vector space V of dimension 2. Let $\mathbb{P}(V)$ denote the projective line consisting of all one-dimensional subspaces of V. Let $\mathrm{SL}(V)$ denote the group of all automorphisms of V that act trivially on the line $\wedge^2 V$. The group of all automorphisms of $\mathbb{P}(V)$

coincides with $\mathrm{PSL}(V) := \mathrm{SL}(V)/(\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z}$ is the center of $\mathrm{SL}(V)$ consisting of $\pm\mathrm{Id}_V$. Note that choosing a basis of V, the Möbius group (the group of fractional linear transformations of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{CP}^1$) gets identified with $\mathrm{PSL}(V)$.

Let X be a Riemann surface. We *do not* assume X to be compact or of finite type. By a *holomorphic coordinate function* on X we will mean a pair of the form (U, ϕ), where $U \subset X$ is some open subset and

$$\phi : U \to \mathbb{P}(V)$$

a biholomorphism of U with the image of ϕ. By a *holomorphic atlas* on X we will mean a collection of holomorphic coordinate functions $\{(U_i, \phi_i)\}_{i\in I}$ such that

$$\bigcup_{i\in I} U_i = X.$$

Let $\{(U_i, \phi_i)\}_{i\in I}$ be a holomorphic atlas satisfying the condition that for each pair $(i, j) \in I \times I$ there is an element $T_{i,j} \in \mathrm{Aut}(\mathbb{P}(V))$ such that the transition function $\phi_i \circ \phi_j^{-1}$ coincides with the restriction of $T_{i,j}$ to $\phi_j(U_i \cap U_j)$.

Another holomorphic atlas $\{(U_j, \phi_j)\}_{j\in J}$ satisfying this condition on transition functions is called *equivalent* to $\{(U_i, \phi_i)\}_{i\in I}$ if the above condition on transition functions holds also for the union $\{(U_k, \phi_k)\}_{k\in I\cup J}$. A *projective structure* on X is an equivalence class of holomorphic atlases satisfying the above condition on transition functions [9].

For our purpose we need a slightly refined structure, which we will call a $\mathrm{SL}(V)$ structure.

A $\mathrm{SL}(V)$ structure on X is defined by giving a holomorphic atlas $\{(U_i, \phi_i)\}_{i\in I}$ together with $A_{i,i'} \in \mathrm{SL}(V)$ for each $(i, i') \in I \times I$ such that

1. the transition function $\phi_i \circ \phi_j^{-1}$ coincides with the restriction to $\phi_j(U_i \cap U_j)$ of the map $A_{i,j} : \mathbb{P}(V) \to \mathbb{P}(V)$;
2. $A_{i,j} = A_{j,i}^{-1}$;
3. $A_{i,j}A_{j,k}A_{k,i} = \mathrm{Id}_V$.

The last two conditions mean that the collection $\{A_{i,j}\}$ form a one-cocycle. Another such data

$$\{\{(U_j, \phi_j)\}_{j\in J}, \{A_{j,k}\}_{j,k\in J}\}$$

satisfying the three conditions is called *equivalent* to it if their union

$$\{\{(U_i, \phi_i)\}_{i\in I\cup J}, \{A_{i,j}\}_{i,j\in I\cup J}\}$$

is a *part* of a data satisfying the above three conditions. A $\mathrm{SL}(V)$ *structure* on X is an equivalence class of such data.

Given a $\mathrm{SL}(V)$ structure $\mathfrak{p}$ on X, any holomorphic coordinate function occurring in any atlas in the equivalence class will be called *compatible* with $\mathfrak{p}$. Note that $\mathfrak{p}$ induces a $\mathrm{SL}(V)$ structure on any open subset U of X in an obvious way.

Clearly, a $\mathrm{SL}(V)$ structure gives a projective structure. The difference between a projective structure and a $\mathrm{SL}(V)$ structure is the following. A $\mathrm{SL}(V)$ structure is a projective structure together with the choice of a *theta characteristic* (see [9]). We recall that a theta characteristic is a holomorphic line bundle $\mathcal{L}$ over X together with a holomorphic isomorphism of $\mathcal{L}^{\otimes 2}$

with the holomorphic cotangent bundle Ω_X. Any Riemann surface has a theta characteristic. If each connected component of X is noncompact, then any line bundle over X, in particular Ω_X, is holomorphically trivializable. For a compact connected Riemann surface of genus g, there are exactly 2^{2g} distinct theta characteristics.

We will show below how a $\mathrm{SL}(V)$ structure defined above gives a theta characteristic. Any Riemann surface admits a projective structure. The uniformization theorem says that the universal cover of a connected Riemann surface Y is biholomorphic to either $\mathbb{C}$ or $\mathbb{CP}^1$ or the upper half plane $\mathbb{H}$. Since the group of all automorphisms of each of these three Riemann surfaces is contained in the Möbius group, Y gets a natural projective structure. The space of all projective structures on Y is an affine space for $H^0(Y, \Omega_Y^{\otimes 2})$, the space of holomorphic quadratic differentials on Y.

Let L_0 denote the tautological line bundle $\mathcal{O}_{\mathbb{P}(V)}(-1)$ over $\mathbb{P}(V)$. Note that $L_0^{\otimes 2}$ is canonically identified with $\Omega_{\mathbb{P}(V)} \otimes \zeta$, where ζ is the trivial line bundle over $\mathbb{P}(V)$ with fiber $\wedge^2 V$. Indeed, for any one-dimensional subspace $\xi \in \mathbb{P}(V)$ of V, we have $L_0|_\xi \cong \xi$ and $\Omega_{\mathbb{P}(V)}|_\xi \cong \mathrm{Hom}(V/\xi, \xi)$. Note that the action of $\mathrm{SL}(V)$ on $\mathbb{P}(V)$ lifts to L_0. Indeed, the standard action of $\mathrm{SL}(V)$ on V gives an action of $\mathrm{SL}(V)$ on L_0.

We fix, once and for all, a nonzero element $\theta \in \wedge^2 V^* \setminus \{0\}$. So θ defines a symplectic structure on V. Using θ, the line bundle $L_0^{\otimes 2}$ gets identified with $\Omega_{\mathbb{P}(V)}$. Indeed, for any line $l \subset V$, the fiber of $\Omega_{\mathbb{P}(V)}$ over the point in $\mathbb{P}(V)$ representing l is canonically identified with $\mathrm{Hom}(V/l, l)$. Since $\mathrm{Hom}(V/l, l) \cong l^{\otimes 2} \otimes \wedge^2 V^*$, the nonzero vector θ identifies $\wedge^2 V^*$ with $\mathbb{C}$, thus identifying $L_0^{\otimes 2}$ with $\Omega_{\mathbb{P}(V)}$.

We will now show how a $\mathrm{SL}(V)$ structure defined above gives a theta characteristic. Let X be equipped with a $\mathrm{SL}(V)$ structure $\mathfrak{p}$. Take a data $\{\{(U_i, \phi_i)\}_{i\in I}, \{A_{i,j}\}_{i,j\in I}\}$, as in the definition of a $\mathrm{SL}(V)$ structure, in the equivalence class for $\mathfrak{p}$. For each $i \in I$, consider the line bundle $\phi_i^* L_0$ on U_i. Since the action $\mathrm{SL}(V)$ on $\mathbb{P}(V)$ lifts to L_0, we can glue $\phi_i^* L_0$ and $\phi_j^* L_0$ over $U_i \cap U_i$ using the clutching function given by the action of $A_{i,j} \in \mathrm{SL}(V)$. Since $\{A_{i,j}\}$ form a one-cocycle, these locally defined line bundles patch together compatibly to define a line bundle over X. Let $\mathcal{L}$ denote the line bundle over X obtained this way.

The isomorphism of $L_0^{\otimes 2}$ with $\Omega_{\mathbb{P}(V)}$ pulls back to an isomorphism of $\phi_i^* L_0^{\otimes 2}$ with $\phi_i^* \Omega_{\mathbb{P}(V)}$ over U_i. Now, the differential $d\phi_i$ gives an isomorphism of $\phi_i^* \Omega_{\mathbb{P}(V)}$ with $\Omega_X|_{U_i}$. Using this and the isomorphism $\phi_i^* L_0 = \mathcal{L}|_{U_i}$ obtained from the construction of $\mathcal{L}$ we get an isomorphism of $\mathcal{L}^{\otimes 2}|_{U_i}$ with $\Omega_X|_{U_i}$. It is evident from the construction that if we consider the similar isomorphism over U_j, then the two isomorphisms of $\mathcal{L}^{\otimes 2}$ with Ω_X over $U_i \cap U_j$ coincide. In other words, we have constructed an isomorphism

$$\Psi : \mathcal{L}^{\otimes 2} \to \Omega_X \tag{2.1}$$

over X. In particular, $\mathcal{L}$ is a theta characteristic. It should be emphasized that $\mathcal{L}$ depends on the $\mathrm{SL}(V)$ structure $\mathfrak{p}$.

Let $O_{\mathcal{L}}$ denote the image of the zero section of $\mathcal{L}$. Let

$$\mathcal{Z} := \mathcal{L} \setminus O_{\mathcal{L}} \tag{2.2}$$

be the complement of the zero section in the total space of $\mathcal{L}$. So $\mathcal{Z}$ is a complex manifold of dimension 2.

Let

$$p_X : \Omega_X \to X$$

denote the natural projection from the total space of the holomorphic cotangent bundle. The complex surface Ω_X has a natural holomorphic symplectic form. Indeed, $p_X^*\Omega_X$ has a tautological section which is defined as follows. Let $dp_X : T\Omega_X \to p_X^* TX$ be the differential of the map p_X, where $T\Omega_X$ (respectively, TX) is the holomorphic tangent bundle of Ω_X (respectively, X). Consider the homomorphism

$$(dp_X)^* : p_X^*\Omega_X \to \Omega_{\Omega_X}.$$

Note that the line bundle $p_X^*\Omega_X$ over the total space of Ω_X has a tautological section that ends any point $z \in \Omega_X$ to z itself. The image of this section by the above homomorphism $(dp_X)^*$ defines a holomorphic one-form ω' on the total space of Ω_X. The exterior derivative

$$\omega_1 := d\omega' \tag{2.3}$$

is a holomorphic symplectic form on Ω_X. This symplectic form ω_1 on the total space of Ω_X can also be described using coordinate charts as follows. Let $U \subset X$ be an open set and $q_0 : U \to \mathbb{C}$ a holomorphic coordinate function on U. The coordinate function q_0 defines a trivialization of the line bundle Ω_X over U. The trivialization sends the constant function 1 to the section dq_0 of the line bundle Ω_X over U. Using this trivialization we get a holomorphic coordinate function (p, q) on the open subset $p_X^{-1}(U) \subset \Omega_X$, where $q = q_0 \circ p_X$; for any $z \in p_X^{-1}(U)$, the evaluation $p(z) \in \mathbb{C}$ satisfies the identity $z = p(z)\, dq_0(p_X(z))$. Now it is easy to see that the restriction to $p_X^{-1}(U)$ of the holomorphic one-form ω' (defined earlier) coincides with $p\,dq$. Therefore, from (2.3) we conclude that $\omega_1 = dp \wedge dq$ over $p_X^{-1}(U)$.

Consider $\mathcal{Z}$ defined in (2.2). Let

$$\Psi_0 : \mathcal{Z} \to \Omega_X \tag{2.4}$$

be the map that sends any z to $\Psi(z \otimes z)$, where Ψ is the homomorphism defined in (2.1). Clearly, Ψ_0 is a degree two étale covering of its image. The pull back

$$\omega := \tfrac{1}{2}(\Psi_0^*\omega_1), \tag{2.5}$$

where ω_1 is defined in (2.3), is a symplectic form on $\mathcal{Z}$.

In [3] it was shown that given a $\mathrm{SL}(V)$ structure on X, there is a natural quantization of $\mathcal{Z}$ equipped with the symplectic structure ω. We will briefly recall the construction of this quantization. First we will recall the definition of a quantization.

2.2. *Quantization of a holomorphic symplectic form*

Let M be a complex manifold. Its holomorphic tangent bundle will be denoted by TM. Let Θ be a holomorphic symplectic form on M. In other words, Θ is a ∂-closed holomorphic two-form on M with the property that for any point $x \in M$, the skew-symmetric bilinear form on the holomorphic tangent space T_xM defined by $\Theta(x)$ is nondegenerate.

Let $\tau : T^*M \to TM$ be the isomorphism defined by the nondegenerate form Θ. So $\tau^{-1}(v)(w) = \Theta(w, v)$, where $v, w \in T_xM$ and $x \in M$.

Let f and g be two holomorphic functions defined on some open subset U on M. Sending the pair (f, g) to $\Theta(\tau(df), \tau(dg))$ defines a *holomorphic Poisson structure* on the space of all locally defined holomorphic functions on M. In other words, the pairing defined by

$$(f, g) \mapsto \{f, g\} := \Theta(\tau(df), \tau(dg)) \tag{2.6}$$

is anticommutative, the Jacobi identity is valid (that is, it defines a Lie algebra structure), and satisfies the Leibniz identity that says $\{fg, f_1\} = g\{f, f_1\} + f\{g, f_1\}$.

Let $\mathcal{H}(M)$ denote the algebra of all (locally defined) holomorphic functions on M. Let $\mathcal{A}(M) := \mathcal{H}(M)[[h]]$ be the space of all formal Taylor series

$$f := \sum_{j=0}^{\infty} h^j f_j,$$

where $f_j \in \mathcal{H}(M)$ and h is a formal parameter.

A *quantization* of the Poisson structure defined in (2.6) is an associative algebra operation on $\mathcal{A}(M)$, which is denoted by $\star$, satisfying the following conditions (see [2,7,8,12] for the details). For any element $g := \sum_{j=0}^{\infty} h^j g_j \in \mathcal{A}(M)$ the product

$$f \star g = \sum_{j=0}^{\infty} h^j \phi_j$$

satisfies the following conditions:

1. each $\phi_i \in \mathcal{H}(M)$ is some polynomial (independent of f and g) in derivatives (of arbitrary order) of $\{f_j\}_{j\geq 0}$ and $\{g_j\}_{j\geq 0}$;
2. $\phi_0 = f_0 g_0$;
3. $1\star f = f\star 1 = f$ for every $f \in \mathcal{H}(M)$;
4. $f\star g - g\star f = \sqrt{-1}h\{f_0, g_0\} + h^2\beta$, where $\beta \in \mathcal{A}(M)$ depends on f, g.

Therefore, $\star$ is a one-parameter deformation of the pointwise multiplication structure on $\mathcal{H}(M)$ with the infinitesimal deformation given by the Poisson structure.

It is known that every C^∞ symplectic structure admits a quantization [7,8]; in fact, every C^∞ Poisson structure admits a quantization [11]. However, in general, there is no natural quantization; the space of all possible C^∞ quantizations of a symplectic structure is infinite dimensional. Equivalence classes of smooth star products on a smooth symplectic manifold are parameterized by sequences with values in the second de Rham cohomology of the manifold [1]. So often it is of interest to be able to give an explicit natural quantization in a given context (see [6,10]).

A constant symplectic structure on a vector space has a canonical quantization, known as the Moyal–Weyl quantization. We will now describe the Moyal–Weyl quantization.

Let V be a complex vector space of dimension $2n$. Let Θ be a constant symplectic form on V. In other words, $\Theta \in \wedge^2 V^*$ defining a nondegenerate skew-symmetric bilinear form on V. Let $\mathcal{H}(V)$ denote the space of all holomorphic functions on V equipped with the Poisson structure defined above.

Let

$$\Delta : V \to V \times V$$

denote the diagonal homomorphism defined by $v \mapsto (v, v)$. There exists a unique differential operator

$$D : \mathcal{H}(V \times V) \to \mathcal{H}(V \times V) \tag{2.7}$$

with constant coefficients such that for any pair $f, g \in \mathcal{H}(V)$,

$$\{f, g\} = \Delta^* D(f \otimes g),$$

where $f \otimes g$ is the function on $V \times V$ defined by $(u, v) \mapsto f(u)g(v)$ [8,12].

The *Moyal–Weyl algebra* is defined by

$$f\star g = \Delta^* \exp(\tfrac{1}{2}(\sqrt{-1}hD))(f \otimes g) \in \mathcal{A}(V) \tag{2.8}$$

for $f, g \in \mathcal{H}(V)$, and it is extended to a multiplication operation on $\mathcal{A}(V)$ using the bilinearity condition with respect to h. In other words, if $f := \sum_{j=0}^{\infty} h^j f_j$ and $g := \sum_{j=0}^{\infty} h^j g_j$ are two elements of $\mathcal{A}(V)$, then

$$f\star g = \sum_{i,j=0}^{\infty} h^{i+j}(f_i\star g_j) \in \mathcal{A}(V).$$

It is known that this $\star$ operation makes $\mathcal{A}(V)$ into an associative algebra that quantizes the symplectic structure Θ. See [2,12] for the details.

Let $\{z_i\}_{1\le i\le 2n}$ be a basis of the dual vector space V^*. So,

$$\Theta = \frac{1}{2}\sum_{i,j=1}^{2n} \omega_{ij} z_i \wedge z_j. \tag{2.9}$$

Let (t_{ij}) be the inverse matrix of the matrix $(-\omega_{ij})_{i,j=1}^{2n}$. So $(1/2)\sum_{i,j=1}^{2n} t_{ij} z_i^* \wedge z_j^*$ is the Poisson structure on V. Let x_i (respectively, y_i) denote the functional on $V \oplus V$ defined by $z_i \circ q_1$ (respectively, $z_i \circ q_2$), where q_j is the projection to the jth factor.

For $f, g \in \mathcal{H}(V)$, the Moyal–Weyl product $f\star g$ has the expression

$$(f\star g)(z) = \sum_{k=0}^{\infty}\left(\frac{1}{k!}\left(\frac{\sqrt{-1}}{2}\sum_{i,j=1}^{2n} t_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j}\right)^k (f(x)g(y))|_{y=x=z}\right)h^k \tag{2.10}$$

(see [8,12]).

Let $\mathrm{Sp}(V)$ denote the group of all linear automorphism of V preserving the symplectic form Θ. The group $\mathrm{Sp}(V)$ acts on $\mathcal{A}(V)$ in an obvious way namely, $(\sum_{j=0}^{\infty} h^j f_j) \circ G = \sum_{j=0}^{\infty} h^j (f_j \circ G)$, where $G \in \mathrm{Sp}(V)$. The differential operator D in (2.7) evidently commutes with the diagonal action of $\mathrm{Sp}(V)$ on $V \times V$. This immediately implies that

$$(f \circ G)\star(g \circ G) = (f\star g) \circ G, \tag{2.11}$$

for any $G \in \mathrm{Sp}(V)$ and $f, g \in \mathcal{A}(V)$.

2.3. SL(V) structure and quantization

Let X be a Riemann surface with a SL(V) structure $\mathfrak{p}$. We will now quantize the symplectic surface $\mathcal{Z}$ defined in Section 2.1.

First set $X = \mathbb{P}(V)$. Note that $\mathbb{P}(V)$ has a tautological SL(V) structure as it can be covered by a single holomorphic coordinate function, namely the identity map. The complex surface $\mathcal{Z}$ (defined in (2.2)) for $X = \mathbb{P}(V)$ will be denoted by $\mathcal{Z}_0$. Clearly, we have

$$\mathcal{Z}_0 = V \setminus \{0\}.$$

The symplectic form ω on $\mathcal{Z}_0$ defined in (2.5) coincides with the restriction of the symplectic form θ on V. (Recall that in Section 2.1 we fixed asymplectic form θ on V.) To see this, let

$$\Psi_{\mathbb{P}(V)} : V \setminus \{0\} \to \Omega_{\mathbb{P}(V)}$$

be the map Ψ_0 (defined in (2.4)) for $\mathbb{P}(V)$. Then the form $\Psi^*_{\mathbb{P}(V)}\omega'$, where ω' as in (2.3), coincides with the contraction $i_{\mathbf{e}}\theta$ with $\mathbf{e}$ being the Euler vector field on $V \setminus \{0\}$ defined by $\mathbf{e}(v) = v$. Finally, since $di_{\mathbf{e}}\theta = 2\theta$, it follows immediately that ω coincides with θ over $V \setminus \{0\}$.

Now, we have the Moyal–Weyl quantization, defined in (2.8), of ω. The identity (2.11) says that the action of SL(V) on $\mathcal{Z}_0 = V \setminus \{0\}$ preserves the quantization.

Now, let X be a general Riemann surface equipped with a SL(V) structure $\mathfrak{p}$. Take a data $\{\{(U_i, \phi_i)\}_{i\in I}, \{A_{i,j}\}_{i,j\in I}\}$, as in the definition of a SL(V) structure, for $\mathfrak{p}$.

Let

$$p : \mathcal{Z} \to X \tag{2.12}$$

be the natural projection. Let

$$p_0 : \mathcal{Z}_0 \to \mathbb{P}(V) \tag{2.13}$$

be the natural projection, that is, the projection p in (2.12) for $X = \mathbb{P}(V)$.

From the construction of $\mathcal{L}$ given in Section 2.1 it follows immediately that the map ϕ_i naturally lifts to a biholomorphism of $p_0^{-1}(\phi_i(U_i))$ with $p^{-1}(U_i)$, where p and p_0 are defined in (2.12) and (2.13), respectively. This biholomorphism takes the holomorphic symplectic form ω on $p^{-1}(U_i)$ to the symplectic form θ on $p_0^{-1}(\phi_i(U_i))$. Indeed, this is an immediate consequence of the construction of the isomorphism Ψ in (2.1) together with the earlier observation that ω for $X = \mathbb{P}(V)$ coincides with θ.

Therefore, using this biholomorphism, the above constructed quantization of the symplectic structure θ on $\mathcal{Z}_0$ gives a quantization of the symplectic manifold $p^{-1}(U_i) \subset \mathcal{Z}$ equipped with the symplectic form ω.

Take any j in the index set I. We noted earlier that the action of SL(V) on $\mathcal{Z}_0$ preserve its quantization. In particular, it is preserved by the action of $A_{i,j}$. Therefore, the two quantizations, namely one on $p^{-1}(U_i)$ and one on $p^{-1}(U_j)$, coincide over $p^{-1}(U_i \cap U_j)$. Consequently, we get a quantization of the symplectic form ω on $\mathcal{Z}$.

3. Properties of quantization

Fix a $\mathrm{SL}(V)$ structure $\mathfrak{p}$ on X. Let $\mathcal{L}$ be the theta characteristic on X associated to $\mathfrak{p}$ (constructed in Section 2.1). The symplectic surface $\mathcal{Z}$ is equipped with a quantization constructed in Section 2.3.

For $i < 0$, by $\mathcal{L}^{\otimes -i}$ we will mean $(\mathcal{L}^*)^{\otimes i}$. By $\mathcal{L}^{\otimes 0}$ we will mean the trivial line bundle.

Take a holomorphic section $s \in H^0(X, \mathcal{L}^{\otimes i})$. This section s defines a holomorphic function Γ_s on $\mathcal{Z}$ as follows. If $i = 0$, then s is simply a holomorphic function on X. In that case, $\Gamma_s = s \circ p$, where p is defined in (2.12). If $i < 0$, then take any $x \in X$ and $v \in p^{-1}(x)$. Now define

$$\Gamma_s(v) := \langle s(x), v^{\otimes -i} \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the contraction of $\mathcal{L}_x^{\otimes i}$ and $\mathcal{L}_x^{\otimes -i}$. If $i > 0$, then define

$$\Gamma_s(v) := (v^*)^{\otimes i}(s(x)),$$

where $v^* \in \mathcal{L}_x^*$ is the dual of v, that is, $v(v^*) = 1$. Note that the linear map

$$\Phi : \bigoplus_{k \in \mathbb{Z}} H^0(X, \mathcal{L}^{\otimes k}) \to H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \tag{3.1}$$

to the space of all holomorphic functions on $\mathcal{Z}$ defined by $\sum_{k \in \mathbb{Z}} u_k \mapsto \sum_{k \in \mathbb{Z}} \Gamma_{u_k}$ is injective, where $\mathcal{O}_{\mathcal{Z}}$ is the sheaf of holomorphic functions on $\mathcal{Z}$. Since

$$\Gamma_s \Gamma_t = \Gamma_{s \otimes t},$$

the image of the map Φ, defined in (3.1), is a subalgebra of $H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$. In other words, the direct sum $\oplus_{k \in \mathbb{Z}} H^0(X, \mathcal{L}^{\otimes k})$ with its natural algebra structure becomes a subalgebra of $H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ using Φ.

Take two sections $s \in H^0(X, \mathcal{L}^{\otimes i})$ and $t \in H^0(X, \mathcal{L}^{\otimes j})$, where $i, j \in \mathbb{Z}$. Let

$$\Gamma_s \star \Gamma_t = \sum_{k=0}^{\infty} h^k \Psi_k \tag{3.2}$$

be the $\star$-product for the quantization corresponding to $\mathfrak{p}$.

Lemma 3.1. *For any $k \geq 0$, there is a holomorphic section $u \in H^0(X, \mathcal{L}^{\otimes (i+j+2k)})$ such that $\Gamma_u = \Psi_k$.*

Proof. Since the homomorphism Φ constructed in (3.1) is injective, it suffices to show that around each point $x \in X$ there is an open neighborhood $U_x \subset X$ such that there is section

$$u_x \in H^0(U_x, \mathcal{L}^{\otimes (i+j+2k)}|_{U_x})$$

with $\Gamma_{u_x} = \Psi_k|_{p^{-1}(U_x)}$, where p is the projection in (2.12). Indeed, in that case, by the injectivity of the map Φ for the Riemann surface $U_x \cap U_y$, two such sections u_x and u_y must coincide over $U_x \cap U_y$. Therefore, these locally defined sections u_x, $x \in X$, patch together compatibly to give a section u of $\mathcal{L}^{\otimes (i+j+2k)}$.

The multiplicative group $\mathbb{C}^*$ acts on $\mathcal{Z}$. The action of $\lambda \in \mathbb{C}^*$ sends any $v \in \mathcal{Z}$ to λv. The quotient of $\mathcal{Z}$ by this action is clearly identified with X. A holomorphic function f on $\mathcal{Z}$ coincides with Γ_α for some holomorphic section

$$\alpha \in H^0(X, \mathcal{L}^{\otimes l})$$

if and only the identity

$$f(\lambda v) = \lambda^{-l} f(v) \tag{3.3}$$

is valid.

Let $\phi : U \to \mathbb{P}(V)$ be a holomorphic coordinate function compatible with the $\mathrm{SL}(V)$ structure $\mathfrak{p}$. Set

$$Z_1 := p_0^{-1}(\phi(U)) \subset \mathcal{Z}_0 = V \setminus \{0\}, \tag{3.4}$$

where p_0, as before, is defined in (2.13).

In Section 2.3 it was noted that from the construction of $\mathcal{L}$ it follows that the map ϕ induces a biholomorphism of $p_0^{-1}(\phi(U))$ with $p^{-1}(U)$, where p is defined in (2.12). Let

$$T : p_0^{-1}(\phi(U)) \to p^{-1}(U)$$

be the biholomorphism obtained from ϕ.

Consider the holomorphic function $f_k := (\Psi_k|_{p^{-1}(U)}) \circ T$ on Z_1 (defined in (3.4)), where T is the biholomorphism defined above, and Ψ_k as in (3.2). In view of (3.3), to prove the lemma it suffices to show that the identity

$$f_k(\lambda v) = \lambda^{-(i+j+2k)} f_k(v) \tag{3.5}$$

is valid for all $v \in Z_1$ and $\lambda \in \mathbb{C}^*$.

Let $s_1 := (\Gamma_s|_{p^{-1}(U)}) \circ T$ and $t_1 := (\Gamma_t|_{p^{-1}(U)}) \circ T$ be the holomorphic functions on Z_1 defined by Γ_s and Γ_t respectively. So, if

$$s_1 \star t_1 = \sum_{l=0}^{\infty} h^j \beta_j,$$

where $\star$ is the Moyal–Weyl quantization of the symplectic structure θ on V, then $\beta_k = f_k$, where f_k is defined above from Ψ_k. Indeed, this is an immediate consequence of the construction of the quantization of $\mathcal{Z}$ done in Section 2.3. The identity (3.3) ensures that

$$s_1(\lambda v) = \lambda^{-i} s_1(v) \quad \text{and} \quad t_1(\lambda v) = \lambda^{-j} t_1(v) \tag{3.6}$$

are valid for all $v \in Z_1$ and $\lambda \in \mathbb{C}^*$.

Take any $\lambda \in \mathbb{C}^*$. Consider the automorphism

$$A_\lambda : V \setminus \{0\} \to V \setminus \{0\}$$

defined by $v \mapsto \lambda v$. Now (3.6) says that

$$s_1 \circ A_\lambda = \lambda^{-i} s_1 \quad \text{and} \quad t_1 \circ A_\lambda = \lambda^{-j} t_1 \tag{3.7}$$

are valid. Also, $A_\lambda^*\theta = \lambda^2\theta$, where θ is the symplectic form on V. Therefore, $A_\lambda^*\theta^* = \lambda^{-2}\theta^*$, where $\theta^* \in \wedge^2 V$ is the dual of θ. Note that θ^* is the Poisson form on V for θ. Therefore, the differential operator in the expression (2.10)

$$D_k := \left(\sum_{i,j=1}^{2} t_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j}\right)^k \tag{3.8}$$

on $V\times V$ has the property that $(A_\lambda \times A_\lambda)^* D_k = \lambda^{-2k} D_k$. Indeed, $\sum_{i,j=1}^2 t_{ij}(\partial/\partial z_i)\otimes(\partial/\partial z_j)$ is the expression of the Poisson operator in terms of the basis $\{z_1, z_2\}$ of V^* (see (2.9)). Therefore, the operator D_k satisfies this condition. From the expression of Moyal–Weyl product in (2.10) it follows that the above identity for D_k and (3.7) together establish (3.5). This completes the proof of the lemma. □

Let

$$\mathcal{I} := \Phi\left(\bigoplus_{k\in\mathbb{Z}} H^0(X, \mathcal{L}^{\otimes k})\right) \subset H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}),$$

where Φ is the injective homomorphism of algebras defined in (3.1). Lemma 3.1 has the following corollary.

Corollary 3.2. *Let X be a Riemann surface with a* $\mathrm{SL}(V)$ *structure. The subspace* $\mathcal{I}[[h]]$ *of* $\mathcal{H}(\mathcal{Z})[[h]]$ *is preserved by the* $\star$*-product on* $\mathcal{H}(\mathcal{Z})[[h]]$ *defining the quantization corresponding to the* $\mathrm{SL}(V)$ *structure.*

In the next section we will consider the differential operators on $\mathcal{Z}$ defined by $\mathcal{I}$.

4. Lifting of symbol of differential operator

4.1. Differential operators and symbol homomorphism

We will briefly recall the definitions of a differential operator and the symbol map.

Let E be a holomorphic vector bundle over a Riemann surface X and n a nonnegative integer. The nth order *jet bundle* of E, denoted by $J^n(E)$, is defined to be the following direct image on X

$$J^n(E) := p_{1*}\left(\frac{p_2^*E}{p_2^*E\otimes \mathcal{O}_{X\times X}(-(n+1)\Delta)}\right),$$

where $p_i : X\times X \to X$, $i = 1, 2$, is the projection onto the ith factor and Δ is the diagonal divisor on $X\times X$ consisting of all points of the form (x, x). There is a natural exact sequence

$$0 \to \Omega_X^{\otimes n}\otimes E \to J^n(E) \to J^{n-1}(E) \to 0 \tag{4.1}$$

which is constructed using the obvious inclusion of $\mathcal{O}_{X\times X}(-(n+1)\Delta)$ in $\mathcal{O}_{X\times X}(-n\Delta)$. The inclusion map $\Omega_X^{\otimes n}\otimes E \to J^n(E)$ is constructed by using the injective homomorphism

$$\Omega_X^{\otimes n} \to J^n(\mathcal{O}_X)$$

which is defined at any $x \in X$ by sending $(df)^{\otimes n} \in (\Omega_X^{\otimes n})_x$, where f is any holomorphic function defined around x with $f(x) = 0$, to the jet of the function $f^n/n!$ at x. Therefore, $J^n(E)$ is a holomorphic vector bundle over X of rank $(n+1)\mathrm{rank}(E)$. Note that $J^0(E) \cong E$.

For another holomorphic vector bundle F over X, the sheaf of *differential operators* of order n from E to F, denoted by $\mathrm{Diff}_X^n(E, F)$, is defined to be

$$\mathrm{Diff}_X^n(E, F) := \mathrm{Hom}_{\mathcal{O}_X}(J^n(E), F) = J^n(E)^* \otimes F.$$

The homomorphism

$$\sigma : \mathrm{Diff}_X^n(E, F) \to \mathrm{Hom}(\Omega_X^{\otimes n} \otimes E, F) \tag{4.2}$$

obtained by restricting a homomorphism $J^n(E) \to F$ to the subbundle $\Omega_X^{\otimes n} \otimes E$ of $J^n(E)$ in (4.1) is known as the *symbol map*.

In particular, $\mathrm{Diff}_X^0(E, F) = \mathrm{Hom}(E, F)$ and the symbol homomorphism on it is the identity map.

4.2. SL(V) *structure and differential operator*

As before, let X be a Riemann surface equipped with a SL(V) structure $\mathfrak{p}$.

For $i, j \in \mathbb{Z}$ and $k \geq 0$, let

$$S(i, j, k) : H^0(X, \mathcal{L}^{\otimes i}) \otimes H^0(X, \mathcal{L}^{\otimes j}) \to H^0(X, \mathcal{L}^{\otimes (i+j+2k)}) \tag{4.3}$$

be the homomorphism that sends any $s \otimes t$ to u constructed in Lemma 3.1.

Take a section $s \in H^0(X, \mathcal{L}^{\otimes i})$. Let

$$S_s^k(j) : H^0(X, \mathcal{L}^{\otimes j}) \to H^0(X, \mathcal{L}^{\otimes (i+j+2k)}) \tag{4.4}$$

be the homomorphism defined by $S_s^k(j)(t) := S(i, j, k)(s \otimes t)$, where $t \in H^0(X, \mathcal{L}^{\otimes j})$. Note that the homomorphism $S(i, j, k)$ in (4.3) is compatible with restrictions to open subsets. In other words, if U is an open subset of X equipped with the SL(V) structure induced by $\mathfrak{p}$, and $S_U(i, j, k)$ is the homomorphism in (4.4) with X replaced by U, then the identity

$$S_U(i, j, k)(s|_U \otimes t|_U) = S(i, j, k)(s \otimes t)|_U \tag{4.5}$$

is valid for all $s \in H^0(X, \mathcal{L}^{\otimes i})$ and $t \in H^0(X, \mathcal{L}^{\otimes j})$.

Since the Moyal–Weyl quantization is expressed in terms of differential operators, the homomorphism $S_s^k(j)$ in (4.4) is given by a differential operator. Since the coefficient of h^k in (2.10) is a differential operator of order k, it follows immediately that $S_s^k(j)$ is given by a differential operator of order at most k from $\mathcal{L}^{\otimes j}$ to $\mathcal{L}^{\otimes (i+j+2k)}$. The identity (4.5) shows that there is a unique differential operator over X giving $S_s^k(j)$ that is compatible with respect to restrictions to open subsets.

If $D \in H^0(X, \mathrm{Diff}^k_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)}))$ is a differential operator, then the symbol of $\sigma(D)$ of D, defined in (4.2), is a holomorphic section of

$$\mathcal{L}^{\otimes(i+j+2k)} \otimes \mathcal{L}^{\otimes -j} \otimes T_X^{\otimes k} = \mathcal{L}^{\otimes i}$$

as $TX = \mathcal{L}^{\otimes -2}$.

Lemma 4.1. *The differential operator of order k*

$$S^k_s(j) \in H^0(X, \mathrm{Diff}^k_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)}))$$

has symbol

$$\sigma(S^k_s(j)) = \frac{(\sqrt{-1})^k}{k!2^k} i(i+1)\cdots(i+k-2)(i+k-1)s = \binom{i+k-1}{k}\left(\frac{\sqrt{-1}}{2}\right)^k s,$$

where $s \in H^0(X, \mathcal{L}^{\otimes i})$ is the section defining $S^k_s(j)$.

Proof. From the construction of the operator $S^k_s(j)$ it follows immediately that it suffices to prove the lemma for $X = U \subset \mathbb{P}(V)$, where U is an open proper subset equipped with the $\mathrm{SL}(V)$ structure induced by the tautological $\mathrm{SL}(V)$ structure on $\mathbb{P}(V)$.

Fix a basis $\{e_1, e_2\}$ of V such that $e_1^* \wedge e_2^* = \theta$, where $\{e_1^*, e_2^*\}$ is the dual basis of V^*. Consider the holomorphic embedding

$$\alpha : \mathbb{C} \to \mathbb{P}(V) \tag{4.6}$$

that sends any $z \in \mathbb{C}$ to the line in V spanned by $ze_1 + e_2$. Let

$$\mathcal{Z}'_0 := p_0^{-1}(\alpha(\mathbb{C})) \subset \mathcal{Z}_0$$

be the inverse image, where p_0 is the projection defined in (2.13). If $X = \alpha(\mathbb{C})$, then $\mathcal{Z} = \mathcal{Z}'_0$.

Now we have a holomorphic isomorphism

$$F : \mathcal{Z}'_0 \to \mathbb{C}^2 \setminus \mathbb{C}$$

defined by

$$F(w_1e_1 + w_2e_2) := \left(\frac{w_1}{w_2}, w_2\right) := (x_1, x_2) \in \mathbb{C}^2 \setminus \mathbb{C}$$

(note that $w_2 \neq 0$ on $\mathcal{Z}'_0$). With this identification F,

$$\theta = x_2\, dx_1 \wedge dx_2$$

over $\mathcal{Z}'_0$, where θ is the symplectic form θ on V.

The projection p_0 (defined in (2.13)) in this identification F is the projection

$$(x_1, x_2) \mapsto x_1.$$

We have

$$F^{-1}(x_1, x_2) = x_1x_2e_1 + x_2e_2 := w_1e_1 + w_2e_2 \in \mathcal{Z}_0',$$

where $w_2(x_1, x_2) = x_2$ and $w_1(x_1, x_2) = x_1x_2$ (note that $x_2 \neq 0$).

We replace X by $\alpha(\mathbb{C})$ equipped with the tautological $\mathrm{SL}(V)$ structure. The identification of $\mathcal{Z}_0'$ with $\mathbb{C}^2 \setminus \mathbb{C}$ using F will be used without any further clarification. So for any holomorphic section

$$\tau \in H^0(\alpha(\mathbb{C}), \mathcal{L}^{\otimes n})$$

over $\alpha(\mathbb{C})$, the function Γ_τ, defined in Section 3, will be considered as a function on $\mathbb{C}^2 \setminus \mathbb{C}$.

Take any section $t \in H^0(\alpha(\mathbb{C}), \mathcal{L}^{\otimes j})$. To calculate the top order term of $S_s^k(j)(t)$ (that is, the action of the symbol of $S_s^k(j)$ on t), we first rewrite the operator D_k on $V \times V$, defined in (3.8), that occurs in the expression for Moyal–Weyl quantization in (2.10), in terms of the coordinates (x_1, x_2) (using F) instead of its original expression in terms of the linear coordinates (w_1, w_2) on $\mathcal{Z}_0' \subset V$. To rewrite, note that

$$\frac{\partial}{\partial w_1} = \frac{1}{x_2}\frac{\partial}{\partial x_1} \quad \text{and} \quad \frac{\partial}{\partial w_2} = \frac{\partial}{\partial x_2} - \frac{x_1}{x_2}\frac{\partial}{\partial x_1}. \tag{4.7}$$

If (w_1, w_2, w_1', w_2') are the coordinates on $V \times V$, where (w_1', w_2') is the copy of (w_1, w_2) in the second factor of $V \times V$, then (4.7) is valid with $w_i, i = 1, 2$, replaced by w_i' and x_i, $i = 1, 2$, replaced by x_i', where (x_1', x_2') is the copy of (x_1, x_2). So the operator $\sum_{i,j=1}^{2} t_{ij}(\partial/\partial x_i)(\partial/\partial y_j)$ in (3.8) is

$$\begin{aligned}\sum_{i,j=1}^{2} t_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j} &= \frac{\partial}{\partial w_1}\frac{\partial}{\partial w_2'} - \frac{\partial}{\partial w_2}\frac{\partial}{\partial w_1'} \\ &= \frac{1}{x_2}\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2'} - \frac{x_1'}{x_2'}\frac{\partial}{\partial x_1'}\right) - \left(\frac{\partial}{\partial x_2} - \frac{x_1}{x_2}\frac{\partial}{\partial x_1}\right)\frac{1}{x_2'}\frac{\partial}{\partial x_1'}\end{aligned}$$

as $t_{ii} = 0$, $i = 1, 2$, and $t_{12} = 1 = -t_{21}$.

Now, on the diagonal of $\mathcal{Z}_0' \times \mathcal{Z}_0'$ we have $x_i = x_i'$, $i = 1, 2$. Therefore,

$$\mathcal{D} := -\frac{1}{x_2}\frac{\partial}{\partial x_1}\frac{x_1'}{x_2'}\frac{\partial}{\partial x_1'} + \frac{x_1}{x_2}\frac{\partial}{\partial x_1}\frac{1}{x_2'}\frac{\partial}{\partial x_1'} = 0 \tag{4.8}$$

when restricted to the diagonal.

Consequently, if $k \geq 1$, to calculate the symbol of the differential operator $S_s^k(j)$ it suffices to consider the kth power of

$$\frac{\partial}{\partial w_1}\frac{\partial}{\partial w_2'} - \frac{\partial}{\partial w_2}\frac{\partial}{\partial w_1'} - \mathcal{D} = \frac{1}{x_2}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2'} - \frac{\partial}{\partial x_2}\frac{1}{x_2'}\frac{\partial}{\partial x_1'} \tag{4.9}$$

where $\mathcal{D}$ is defined in (4.8).

We can further simplify the computation of the symbol as follows. We will show that the term

$$\frac{1}{x_2}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2'} \tag{4.10}$$

in the right-hand side of (4.9) does not contribute to the symbol. To prove this assertion, fix $x_1 = a$ and let

$$S_a := \{(a, x) | x \in \mathbb{C}^*\} \subset \mathcal{Z}'_0$$

be the subset. We noted earlier that the projection p_0 defined in (2.13) is the projection of $\mathbb{C}^2 \setminus \mathbb{C}$ to the first factor. Hence S_a is a fiber of the projection p_0. Consequently, for $t \in H^0(\alpha(\mathbb{C}), \mathcal{L}^{\otimes j})$, the restriction to S_a of the function

$$\hat{t} := \Gamma_t$$

on $\mathcal{Z}'_0$ (defined in Section 3) depends only on the evaluation of the section t at the point $\alpha(a)$, where α is defined in (4.6). So, the restriction of the function $\partial \hat{t}/\partial x_2$ to S_a depends only on $t(\alpha(a))$. Therefore, if $k \geq 1$, then the term (4.10) does not contribute to the symbol.

Hence the symbol of the differential operator $S_s^k(j)$ coincides with the symbol of the differential operator

$$\mathcal{D}' := \frac{1}{k!} \left(-\frac{\sqrt{-1}}{2} \frac{\partial}{\partial x_2} \frac{1}{x'_2} \frac{\partial}{\partial x'_1} \right)^k \Bigg|_{x_2 = x'_2} \tag{4.11}$$

(see the coefficient of h^k in (2.10)).

If $u \in H^0(\alpha(\mathbb{C}), \mathcal{L}^{\otimes l})$ and $\hat{u} := \Gamma_u$ the corresponding function on $\mathcal{Z}'_0 = \mathbb{C}^2 \setminus \mathbb{C}$, then

$$\frac{\partial \hat{u}}{\partial x_2} = -\frac{l}{x_2} \hat{u}.$$

Indeed, this follows immediately from the identity (3.3). Using this, the symbol of the differential operator $\mathcal{D}'$ defined in (4.11) is

$$\sigma(\mathcal{D}') = \frac{1}{k!(2\sqrt{-1})^k} (-i)(-i-1) \cdots (-i-k+1) s.$$

Note the symbol is defined in (4.2) in such a way that the differential operator $\mathrm{d}^n/\mathrm{d}x^n$ on $\mathbb{C}$ is 1.

Since $\sigma(\mathcal{D}') = \sigma(S_s^k(j))$, the proof of the lemma is complete. □

In the next section we will use Lemma 4.1 to decompose a differential operator using the symbol homomorphism.

5. Decomposition of a differential operator

As in the previous section, let X be a Riemann surface equipped with a $\mathrm{SL}(V)$ structure $\mathfrak{p}$. Consider a differential operator

$$\mathcal{D}_0 \in H^0(X, \mathrm{Diff}_X^k(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes (i+j+2k)})) \tag{5.1}$$

of order $k \geq 1$, with $i \notin [-2(k-1), 0]$.

So, the symbol $\sigma(\mathcal{D}_0)$ of the operator in (5.1) is a section of $\mathcal{L}^{\otimes i}$ (see (4.2)). If $\mathcal{D}_0$ is exactly of order k then $\sigma(\mathcal{D}_0) \neq 0$. Set

$$s_0 := \sigma(\mathcal{D}_0) \binom{i+k-1}{k}^{-1} \frac{k!2^k(\sqrt{-1})^k}{(\sqrt{-1})^k}.$$

Note that

$$\binom{i+k-1}{k} \neq 0$$

as the condition $i \notin [-2(k-1), 0]$ implies that $i \notin [-k+1, 0]$. From Lemma 4.1 it follows immediately that $\sigma(\mathcal{D}_0) = \sigma(S^k_{s_0}(j))$. Consequently,

$$\mathcal{D}_1 := \mathcal{D}_0 - S^k_{s_0}(j)$$

is a differential operator of order at most $k-1$. Assume that $k-1 \geq 1$.

Now we repeat the above construction by replacing k by $k_1 = k-1$ and i by $i_1 = i+2$. Note that the initial assumption $i \notin [-2(k-1), 0]$ and the assumption $k_1 \geq 1$ together ensure that

$$\binom{i_1+k_1-1}{k_1} \neq 0,$$

that is, $i_1 \notin [-k_1+1, 0]$. Therefore, the construction can be repeated. So we get a holomorphic section

$$s_1 \in H^0(X, \mathcal{L}^{\otimes(i+2)}) = H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega_X)$$

and a differential operator

$$\mathcal{D}_2 \in H^0(X, \mathrm{Diff}^{k-2}_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)}))$$

of order $k-2$ in place of s_0 and $\mathcal{D}_1$, respectively.

Now we can proceed inductively as follows. For the differential operator

$$\mathcal{D}_l \in H^0(X, \mathrm{Diff}^{k-l}_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)}))$$

of order $k-l$, if $k-l \geq 1$, then set $i' = i+2l$, $k' = k-l$ and repeat the initial construction by substituting i by i', k by k' and $\mathcal{D}_0$ by $\mathcal{D}_l$. The initial assumption $i \notin [-2(k-1), 0]$ implies that $i' \notin [-k'+1, 0]$. The section obtained in place of s_0 will be denoted by s_l and the differential operator obtained in place of $\mathcal{D}_1$ will be denoted by $\mathcal{D}_{l+1}$. Note that

$$s_l \in H^0(X, \mathcal{L}^{\otimes(i+2l)}) = H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega^{\otimes l}_X)$$

and $\mathcal{D}_{l+1}$ is a differential operator of order $k-l-1$. The initial assumption $i \notin [-2(k-1), 0]$ ensures that this inductive process can be repeated until we get a differential operator of order 0.

Therefore, we have a homomorphism

$$S : H^0(X, \mathrm{Diff}^k_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)})) \to \bigoplus_{l=0}^{k} H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega_X^{\otimes l}) \tag{5.2}$$

that sends any differential operator $\mathcal{D}_0$ to $\sum_{l=0}^{k-1} s_l$. Recall that a zeroth-order differential operator is a vector bundle homomorphism with the symbol homomorphism being the identity map.

Note that from the above construction of S it follows immediately that

$$S(H^0(X, \mathrm{Diff}^{k-l'}_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)}))) \subset \bigoplus_{l=l'}^{k} H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega_X^{\otimes l})$$

for each $l' \in [0, k]$.

Define the homomorphism

$$S' : \bigoplus_{l=0}^{k} H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega_X^{\otimes l}) \to H^0(X, \mathrm{Diff}^k_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)})) \tag{5.3}$$

that sends any

$$\sum_{l=0}^{k} s_l \in \bigoplus_{l=0}^{k} H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega_X^{\otimes l})$$

to the differential operator $\sum_{l=0}^{k} S^{k-l}_{s_l}(j)$, where $S^{k-l}_{s_l}(j)$ are as in Lemma 4.1.

The above homomorphism S' has the property that

$$S'\left(\bigoplus_{l=l'}^{k} H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega_X^{\otimes l})\right) \subset H^0(X, \mathrm{Diff}^{k-l'}_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)}))$$

for each $l' \in [0, k]$. Indeed, this is an immediate consequence of the fact that $S^{k-l}_{s_l}(j)$, $l \in [0, k]$, is a differential operator of order $k - l$ (see Lemma 4.1).

From the construction of the homomorphism S in (5.2) it is evident that S and S' are inverses of each other, where S' is defined in (5.3).

The above constructions are put down in the form of the following theorem.

Theorem 5.1. *Let X be a Riemann surface equipped with a* $\mathrm{SL}(V)$ *structure. If $i \notin [-2(k-1), 0]$, then the homomorphism*

$$S : H^0(X, \mathrm{Diff}^k_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)})) \to \bigoplus_{l=0}^{k} H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega_X^{\otimes l})$$

constructed in (5.2) *is an isomorphism with S' defined in* (5.3) *being its inverse. Furthermore, for any integer $l' \in [0, k]$,*

$$S(H^0(X, \mathrm{Diff}^{k-l'}_X(\mathcal{L}^{\otimes j}, \mathcal{L}^{\otimes(i+j+2k)}))) \subset \bigoplus_{l=l'}^{k} H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega_X^{\otimes l}).$$

In Theorem 5.1, if we set $j = 1, k = 1$ and $i = 0$ (this is *not* allowed due to the condition $i \notin [-2(k-1), 0]$), then the right-hand and the left-hand sides of the homomorphism S are

$$H^0(X, \mathcal{O}_X) \oplus H^0(X, \Omega_X) \quad \text{and} \quad H^0(X, \mathrm{Diff}^1_X(\mathcal{L}, \mathcal{L}^{\otimes 3})),$$

respectively. We will show that there is no differential operator on X from $\mathcal{L}$ to $\mathcal{L}^{\otimes 3}$ with the constant function 1 as its symbol provided X is a compact connected Riemann surface of genus at least two. Indeed, any such differential operator D gives a splitting

$$\begin{tikzcd} 0 \arrow[r] & \Omega_X \otimes \mathcal{L} \arrow[r] & J^1(\mathcal{L}) \arrow[r] \arrow[d, "D"] & \mathcal{L} \arrow[r] & 0 \\ & \mathcal{L}^{\otimes 3} = & \Omega_X \otimes \mathcal{L} & & \end{tikzcd}$$

of the jet sequence where the exact sequence is the one in (4.1) for $E = \mathcal{L}$ and $n = 1$. The vector bundle $J^1(\mathcal{L})$ admits flat connection. In fact, any projective structure on X induces a flat connection on $J^1(\mathcal{L})$ [9]. Therefore, if X is a compact Riemann surface of genus at least two, then the above jet sequence does not split (as any holomorphic direct summand of a flat vector bundle must be of degree 0 but the degree of $\mathcal{L}$ is genus$(X) - 1$).

The above example shows that the isomorphism S in (5.2) cannot be extended as an isomorphism (for dimensional reasons) for all k and i.

However, the allowed range $i \notin [-2(k-1), 0]$ in Theorem 5.1 can be marginally expanded, which will be explained below.

Set $j = -k$ and $i = 2$ in Theorem 5.1. So we have

$$H^0(X, \mathrm{Diff}^k_X(\mathcal{L}^{\otimes -k}, \mathcal{L}^{\otimes (k+2)})) \cong \bigoplus_{l=1}^{k+1} H^0(X, \Omega_X^{\otimes l}). \tag{5.4}$$

Although, Theorem 5.1 does not allow $i = 0$, the decomposition in (5.4) extends to a decomposition

$$H^0(X, \mathrm{Diff}^{k+1}_X(\mathcal{L}^{\otimes -k}, \mathcal{L}^{\otimes (k+2)})) \cong \bigoplus_{l=0}^{k+1} H^0(X, \Omega_X^{\otimes l}). \tag{5.5}$$

The above isomorphism is constructed as follows. Given a $\mathrm{SL}(V)$ structure on X, in [4] a differential operator

$$\mathcal{D}(k+1) \in H^0(X, \mathrm{Diff}^{k+1}_X(\mathcal{L}^{\otimes -k}, \mathcal{L}^{\otimes (k+2)}))$$

was constructed whose symbol is the constant function 1 (see [4, Theorem 4.1, p. 465]). Given a differential operator

$$D \in H^0(X, \mathrm{Diff}^{k+1}_X(\mathcal{L}^{\otimes -k}, \mathcal{L}^{\otimes (k+2)})),$$

its symbol $\sigma(D)$ is a holomorphic function on X. The differential operator

$$D - \sigma(D)\mathcal{D}(k+1) \in H^0(X, \mathrm{Diff}^k_X(\mathcal{L}^{\otimes -k}, \mathcal{L}^{\otimes (k+2)})).$$

Consequently, the decomposition (5.4) for the differential operator $D - \sigma(D)\mathcal{D}(k+1)$ gives the decomposition (5.5) for D. In other words, if

$$D - \sigma(D)\mathcal{D}(k+1) = \sum_{l=1}^{k+1} \theta_l$$

by (5.4), where θ_l is a holomorphic section of $\Omega_X^{\otimes l}$, then D corresponds to $\sigma(D) + \sum_{l=1}^{k+1} \theta_l$ by the isomorphism in (5.5).

Remark 5.2. Let W be a holomorphic vector bundle over X equipped with a holomorphic connection ∇. Since any holomorphic connection on a Riemann surface is flat, the connection ∇ induces a canonical isomorphism of vector bundles

$$J^i(W \otimes E) \cong W \otimes J^i(E) \tag{5.6}$$

for any $i \geq 0$, where E is any holomorphic vector bundle. The isomorphism in (5.6) can be constructed as follows. For any $x \in X$ and $w \in W_x$, let $\hat{w}$ be the (unique) flat section of W, defined around x, such that $\hat{w}(x) = w$. Now for any holomorphic section s of E defined around x, send the pair (w, s) to the holomorphic section $\hat{w} \otimes s$ of $W \otimes E$ defined around x. Consider the element in $(J^i(W \otimes E))_x$ defined by $\hat{w} \otimes s$. If s vanishes at x of order $i+1$, then clearly $\hat{w} \otimes s$ also vanishes at x of order $i+1$. Therefore, $(w, s) \mapsto \hat{w} \otimes s$ induces a holomorphic homomorphism of vector bundles $W \otimes J^i(E) \to J^i(W \otimes E)$. It is straight-forward to check that this homomorphism is an isomorphism.

Using the isomorphism in (5.6), Theorem 5.1 gives

$$H^0(X, \mathrm{Diff}_X^k(\mathcal{L}^{\otimes j} \otimes W, \mathcal{L}^{\otimes(i+j+2k)} \otimes E)) \cong \bigoplus_{l=0}^{k} H^0(X, \mathcal{L}^{\otimes i} \otimes \Omega_X^{\otimes l} \otimes \mathrm{Hom}(W, E))$$

where i, j, k are as in Theorem 5.1. Similarly, (5.5) gives

$$H^0(X, \mathrm{Diff}_X^{k+1}(\mathcal{L}^{\otimes -k} \otimes W, \mathcal{L}^{\otimes(k+2)} \otimes E)) \cong \bigoplus_{l=0}^{k+1} H^0(X, \Omega_X^{\otimes l} \otimes \mathrm{Hom}(W, E)).$$

The notion of projective structure on a Riemann surface can be generalized to higher dimensions as follows. Let V be a complex vector space of dimension $2d+2$ equipped with a nondegenerate anti-symmetric bilinear form $\theta \in \wedge^2 V^*$. Let $\mathrm{Sp}(V)$ denote the group of all linear automorphisms of V preserving the symplectic form θ. So $G := \mathrm{Sp}(V)/(\mathbb{Z}/2\mathbb{Z})$ is a simple group. Let $P(V)$ denote the projective space parameterizing all lines in V. So G acts faithfully on $P(V)$. Let M be a complex manifold of dimension $2d+1$. A holomorphic coordinate function on M is a pair (U, ϕ), where $U \subset M$ is an open subset and $\phi : U \to P(V)$ is a holomorphic embedding. A *G-structure* on M is defined by giving a covering of M by holomorphic coordinate functions $\{(U_i, \phi_i)\}_{i \in I}$ such that each transition function $\phi_i \circ \phi_i^{-1}$ is the restriction of some automorphism $T_{j,i} \in G \subset \mathrm{Aut}(P(V))$ to $\phi_i(U_i \cap U_j)$. So if $d = 0$, then a G-structure is a projective structure on a Riemann surface.

Let M be a complex manifold of dimension $2d+1$ equipped with a G-structure. The G-structure defines a holomorphic contact structure $F \subset TM$ on M, where TM is the holomorphic tangent bundle of M (see [5]). Let $N := (TM/F)^*$ be the holomorphic line bundle over M. We will denote by N' the complement of the zero section in the total space of N. (Note that N' is naturally identified with the complement of the zero section in the dual line bundle N^*.) The complex manifold N' has a natural holomorphic symplectic structure. In [5], a canonical quantization of this symplectic manifold was constructed (see [5, Theorem 4.2, p. 365]). As in the case of $d = 0$, this quantization is constructed using the Moyal–Weyl quantization of the symplectic structure θ on V.

Given a holomorphic section s of $N^{\otimes i}$, $i \in \mathbb{Z}$, as before, we have holomorphic function Γ_s on N' that satisfies the identity $\Gamma_s(v) = \langle s(x), v^{\otimes -i}\rangle$, where $x \in M$ is the projection of v. For another holomorphic section $t \in H^0(M, N^{\otimes j})$, consider

$$\Gamma_s \star \Gamma_t = \sum_{k=0}^{\infty} h^k \Psi_k,$$

where the $\star$-product is with respect to the canonical quantization of N' constructed in [5]. Now it is easy to check that the following version of Lemma 3.1 is valid.

Modified version of Lemma 3.1. *For any $k \geq 0$, there is a unique holomorphic section*

$$u \in H^0(M, N^{\otimes(i+j+k)})$$

such that $\Psi_k = \Gamma_u$.

Therefore, as in (4.4), sending any t to u constructed above we obtain a holomorphic differential operator

$$S_s^k(j) \in H^0(M, \mathrm{Diff}_M^k(N^{\otimes j}, N^{\otimes(i+j+k)}))$$

of order k on M.

The symbol $\sigma(S_s^k(j))$ of the differential operator $S_s^k(j)$ is a holomorphic section of $N^{\otimes(i+k)} \otimes \mathrm{Sym}^k TM$ over M, where $\mathrm{Sym}^k TM$ denotes the kth symmetric power of the tangent bundle.

The symmetric power of the natural projection of TM to N^* defines a homomorphism

$$f : \mathrm{Sym}^k TM \to (N^*)^{\otimes k}.$$

Now let s' be the holomorphic section of $N^{\otimes i}$ defined by the image of $\sigma(S_s^k(j))$ in the following composition of homomorphisms:

$$\begin{aligned} \sigma(S_s^k(j)) \in H^0(M, N^{\otimes(i+k)} \otimes \mathrm{Sym}^k TM) &\xrightarrow{f} H^0(M, N^{\otimes(i+k)} \otimes (N^*)^{\otimes k}) \\ &\cong H^0(M, N^{\otimes i}), \end{aligned} \tag{5.7}$$

where $N^{\otimes(i+k)} \otimes (N^*)^{\otimes k} \to N^{\otimes i}$ is the contraction of $N^{\otimes k}$ with its dual $(N^*)^{\otimes k}$.

As done in Lemma 4.1, it can be shown that the above constructed section $s' \in H^0(M, N^{\otimes i})$ is a constant scalar multiple of the section s that we started with.

In other words, using the quantization we are able to recover the image of $\sigma(S_s^k(j))$ in $H^0(M, N^{\otimes i})$ (for the composition homomorphism in (5.7)), but not $\sigma(S_s^k(j))$ itself. For example, this construction of a differential operator from a symbol does not give a nontrivial differential operator if the symbol comes from the kernel of the composition homomorphism in (5.7).

The key point in the decomposition of differential operators constructed in Theorem 5.1 is to be able to construct a differential operator from a given section of a tensor power of $\mathcal{L}$ with the property that the symbol coincides with the given section. As we have shown above, if $d \geq 1$ then from a given section of a tensor power of N we are still able to construct a differential operator, but the symbol of the differential operator in general does not coincide with the given section. Consequently, Theorem 5.1 and (5.5) proved for $d = 0$ do not generalize to $d \geq 1$.

References

[1] M. Bertelson, M. Cahen, S. Gutt, Equivalence of star-products, Class. Quant. Grav. 14 (1997) 93–107.
[2] F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz, D. Sternheimer, Deformation theory and quantizations, I and II, Ann. Phys. 111 (1978) 61–151.
[3] D. Ben-Zvi, I. Biswas, A quantization on Riemann surfaces with projective structure, Lett. Math. Phys. 54 (2000) 73–82.
[4] I. Biswas, A remark on the jet bundles over the projective line, Math. Res. Lett. 3 (1996) 459–466.
[5] I. Biswas, R. Dey, Quantization and contact structure on manifolds with projective structure, J. Geom. Phys. 42 (2002) 355–369.
[6] M. Cahen, S. Gutt, J. Rawnsley, On tangential star product for the coadjoint Poisson structure, Comm. Math. Phys. 180 (1996) 99–108.
[7] M. De Wilde, P.B.A. Lecomte, Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifold, Lett. Math. Phys. 7 (1983) 487–496.
[8] B.V. Fedosov, A simple geometrical construction of deformation quantization, J. Diff. Geom. 40 (1994) 213–238.
[9] R.C. Gunning, Lectures on Riemann surfaces, Mathematical Notes 2, Princeton University Press, Princeton, NJ, 1966.
[10] S. Gutt, An explicit ★-product on the cotangent bundle of a Lie group, Lett. Math. Phys. 7 (1983) 249–258.
[11] M. Kontsevich, Deformation quantization of Poisson manifolds, I, Preprint. http://xxx.lanl.gov/abs/q-alg/9709040.
[12] A. Weinstein, Deformation quantization. Séminaire Bourbaki, no. 789, 46ème année, 1994, pp. 213–238.